no. 1.
May 26. 1748.

Honoured Sir, + ~~[illegible]~~

I take ye Liberty to offer the Paper, which comes with this, to your Perusal; and should be glad, if it meets with your Approbation, that you would lay it before ye Society; which will be esteemed a Great Favour by,

Hond. Sir,

your most humble and obedient Servant

T. Simpson

Woolwich. May. 18. 1748.

To Wm. Jones Esq;

Po/ve/k

XI. Of the Fluents of Multinomials and Series affected by radical-Signs, which do not begin to converge 'till after ye 2d. Term – In a Letter to Wm. Jones Esqr. F.R.S. by T. Simpson. F.R.S.

Presented [Ma]y 5 1748

Although ye Application of Infinite Series and the Quadrature of ye Conic-Sections, to ye Inverse Method of Fluxions has exercised ye Pens of ye most able Mathematicians and produced many curious and useful Discoveries; yet Nothing has been hitherto given, that I know of, whereby the Fluents of radical Multinomials and Series, which do not begin to converge till after ye 2d. Term, can be determined, so as to be of use in ye Solution of Problems: The common Method, by expanding the given Expression being, you know, altogether [illegible] impracticable in this Case.

The Consideration of which, ~~with the frequent Occasion I have met with something of this Nature, first~~ induced me to draw up ye following Paper, which I humbly beg Leave to lay before you, who are so good a Judge of ye various Improvements which ye Subject has from time to time, received.

What most encourages me to hope this little Essay will meet with your Approbation is, that it is not meerly an abstracted useless Speculation, but may be apply'd to good Purpose in many difficult and important Enquiries into Nature: Whereof I have put down one or two Instances, and shall further take ye Liberty to observe here, that most of ye Lunar Equations given by Sir Isaac Newton are, only, such Approximations as may be exhibited by ye first Term of a Series, derived by ye Method here delivered.

Proposition

The Fluent of $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$ being given (either in algebraic Terms, or from y^e Quadrature of y^e Conic-Sections &c.) tis proposed, by Means thereof, to approximate the Fluent of $\overline{a+cx^n+dx^{2n}+ex^{3n}+fx^{4n}\,\&c.}|^m \times x^{pn-1}\dot{x}$; supposing the Series not to converge 'till after y^e 2^d. Term.

Make $cz^n = cx^n + dx^{2n} + ex^{3n}$ &c. and let Q be y^e given Fluent of $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$, answering to any proposed Value of x: Moreover let $y = x^{pn}$, or $y^{\frac{1}{p}} = x^n$, and let this Value of x^n be substituted in y^e first Equation, and it will become $cz^n = cy^{\frac{1}{p}} + dy^{\frac{2}{p}} + ey^{\frac{3}{p}}$ &c. Whereof y^e Root (y) being extracted, we shall (by making $R = -\frac{pd}{c}$, $S = \frac{p.\,p+3}{2}\times\frac{d^2}{c^2} - \frac{pe}{c}$, $T = -\frac{p.\,p+4.\,p+5}{6}\times\frac{d^3}{c^3} + \frac{p.\,p+4}{1}\times\frac{de}{c^2} - \frac{pf}{c}$ &c.) have $y\,(x^{pn}) = z^{pn} + Rz^{pn+n} + Sz^{pn+2n}$ &c. whence we also obtain

$$x^{pn-1}\dot{x} = z^{pn-1}\dot{z} + \frac{p+1}{p}\times Rz^{pn+n-1}\dot{z} + \frac{p+2}{p}\times Sz^{pn+2n-1}\dot{z}\ \&c.$$

Let this Value, with that of $cx^n + dx^{2n} + ex^{3n}$ &c. (above given) be now substituted in y^e proposed Fluxion, and it will become

$$\overline{a+cz^n}|^m \times \overline{z^{pn-1}\dot{z} + \frac{p+1}{p}\times Rz^{pn+n-1}\dot{z} + \frac{p+2}{p}\times Sz^{pn+2n-1}\dot{z}\ \&c.}$$

Moreover, let v denote y^e Place, or distance of any Term, of this Expression, from y^e first (exclusive) then y^e Term it self (Drawn into y^e Common Multiplicator) will be denoted by $\overline{a+cz^n}|^m \times \frac{p+v}{p}\times Az^{pn+vn-1}\dot{z}$; and the Fluent thereof will be truly expressed by $\frac{p+1}{p+m+1}\times\frac{p+2}{p+m+2}\times\frac{p+3}{p+m+3}\times\ldots\frac{p+v}{p+m+v}\times\overline{\frac{a}{-c}}|^v \times AQ$, $+\ \frac{p+v.\,A}{p}\times\frac{\overline{a+cz^n}|^{m+1}\times z^{pn-n}}{p+m+v\times nc}$ into $z^{vn} - \frac{p+v-1}{p+m+v-1}\times\frac{az^{vn-n}}{c} + \frac{p+v-1.\,p+v-2}{p+v+m-1.\,p+v+m-2}\times \frac{a^2z^{vn-2n}}{c^2}$ &c. continued to as many Terms as there are Units in v. Wherein let v be expounded by 1, 2, 3 &c. successively, and R, S, T &c. by A respectively: By which Means y^e Fluent of y^e whole Expression will be obtained.

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Corol. 1

Because ye Fluent of ye General Term, when ye Multiplicator $\overline{a+cz^n}|^{m+1}$ becomes $=0$, is barely $=\frac{p+1}{p+m+1}\times\frac{p+2}{p+m+2}\times\frac{p+3}{p+m+3}\times\ldots\frac{p+v}{p+m+v}\times\overline{\frac{a}{-c}}|^{v}\times AQ$, the Fluent of ye Whole Expression will, therefore, in this Case, be truly defined by

$$Q\times\overline{1-\frac{p+1\cdot Ra}{p+m+1\cdot c}+\frac{p+1\cdot p+2\cdot Sa^2}{p+m+1\cdot p+m+2\cdot c^2}-\frac{p+1\cdot p+2\cdot p+3\cdot Ta^3}{p+m+1\cdot p+m+2\cdot p+m+3\cdot c^3}}\ \&c$$

Where Q denotes ye Fluent of $\overline{a+cz^n}|^{m}\times z^{pn-1}\dot{z}$, when $z^n=\frac{a}{-c}$.

Corol. 2

But, if $m+1$ and p be, each of them, the Half of an odd affirmative Number, and P be taken to denote ye Periphery of a Circle whose Diameter is Unity, and $-c$ be put $=b$, then the Value of Q (or ye Fluent of $\overline{a-bz^n}|^{m}\times z^{pn-1}\dot{z}$, when $z^n=\frac{a}{b}$) will be $=\frac{a^{p+m}P}{nb^{p}}\times\frac{1.3.5.7\,\&c.\,(\text{to } p-\frac{1}{2} \text{ Factors})\times 1.3.5.7\,\&c.\,(\text{to } m+\frac{1}{2} \text{ Factors})}{2.4.6.8.10.12\,\&c.\,(\text{to } p+m \text{ Factors})}$

Therefore ye whole required Fluent ~~will, in this Case be~~ of $\overline{a-bx^n+dx^{2n}+ex^{3n}\,\&c}|^{m}\times x^{pn-1}\dot{x}$ is, in this Case, equal to ye Product of that Expression into ye following Series, $1+\frac{p+1\cdot Ra}{p+m+1\cdot b}+\frac{p+1\cdot p+2\cdot Sa^2}{p+m+1\cdot p+m+2\cdot b^2}\,\&c.$ Wherein R is to be taken $=\frac{pd}{b}$, $S=\frac{p\cdot p+3}{2}\times\frac{d^2}{b^2}+\frac{pe}{b}$, $T=\frac{p\cdot p+4\cdot p+5}{6}\times\frac{d^3}{b^3}+\frac{p\cdot p+4}{1}\times\frac{de}{b^2}+\frac{pf}{b}$, &c. according to what is above specified.

The Use of what has been delivered above will, in some Measure, appear from ye Solution of ye two following Problems, which I shall subjoin as Examples thereof.

The first is; To find the Time of Oscillation in ye Arch of a Cycloid, in a Medium resisting according to ye duplicate Ratio of ye Velocity.

Let A denote ye whole Arch of ye Semi-Cycloid, or ye Length of ye Pendulum, a the Arch described in the whole

ye Descent, and x any variable part thereof described from ye Beginning of ye Descent; and let ye Density of ye Medium be, every where, as $\frac{1}{h}$: Then ye Fluxion of ye Time will be found =

$$\overline{a - \overline{1+\frac{2a}{h}} \times \frac{x}{2} - \frac{2x^2}{2.3h} + \frac{4x^3}{2.3.4h^2} - \frac{8x^4}{2.3.4.5h^3} \&c.}\Big|^{-\frac{1}{2}} \times \overline{2A}|^{\frac{1}{2}} \times x^{-\frac{1}{2}}\dot{x}\ ^*$$

: which being compared with $\overline{a - bx^n + dx^{2n} + ex^{3n} \&c.}|^m \times x^{pn-1}\dot{x}$ (vid. Corol. 2) we shall, in this Case, have $n=1$, $m=-\frac{1}{2}$, $p=\frac{1}{2}$, $a=a$, $b=1+\frac{2a}{h}\times\frac{1}{2}$, $\frac{d}{b}=\frac{2}{3h}$, $\frac{e}{b}=-\frac{1}{3h^2}$, $\frac{f}{b}=\frac{2}{15h^3}$ &c. Whence $R=\frac{1}{3h}$, $S=\frac{2}{9h^2}$, $T=\frac{8}{45h^3}$ &c.

also, $\frac{a^{p+m}P}{nb^p} \times \frac{1.3.5.7(p-\frac{1}{2}) \times 1.3.5.7(m+\frac{1}{2})}{2.4.6.8.10(p+m)} = \frac{P}{b^{\frac{1}{2}}}$, and

$$1 + \frac{p+1.Ra}{p+m+1.b} + \frac{p+1.p+2.Sa^2}{p+m+1.p+m+2.b^2} \&c. = 1 + \frac{a}{2bh} + \frac{5a^2}{12b^2h^2} + \frac{7a^3}{18b^3h^3} \&c.$$ Rx 333

Whence we have $\overline{2A}|^{\frac{1}{2}} \times \frac{P}{b^{\frac{1}{2}}} \times \overline{1 + \frac{a}{2bh} + \frac{5a^2}{12b^2h^2} \&c.}$, for ye Time of one Whole Vibration of ye Pendulum; which by substituting $\overline{1+\frac{2a}{h}\times\frac{1}{2}}$ for its Equal b, &c. becomes $PA^{\frac{1}{2}} \times \overline{1 \ * + \frac{a^2}{6h^2} - \frac{2a^3}{9h^3} \&c.}$

From which it appears that ye Effect of ye Resistance on ye Time of Vibration, in small Arches, is nearly in ye Duplicate Ratio of those Arches.

Sir Isaac Newton (from whom it is impossible to disagree without being under some Apprehensions of a Mistake) has, indeed, given a very different Solution to this Problem (in Princip. Prop. 27. B. 2) But as ye Conclusion here brought out exactly agrees with what I have elsewhere given, by a different Method, I have great Reason to believe I have no where falln into an Error.

The 2d. Example I shall give as an Illustration of ye foregoing Method is, To determine ye Apside Angle (or ye Angle of ye 2 Apses at ye Center) in an Orbit described by Means of a Centripetal Force which varies according to any Power of ye Distance.

In order to which, let ye Velocity of ye Body at the higher Apse

* The Investigation of this, and ye Fluxion in ye following Example, are both given in my Essays

be to that whereby it might describe a Circle at ye same Distance from ye Center, in ye given Ratio of p to Unity; also let that Distance be denoted by Unity [redacted]; and, supposing z to denote any other Distance, let ye Centripetal Force be universally expressed by z^n. Then ye Fluxion of ye Angle at ye Center will be expressed by $\frac{-p\dot{z}}{z\sqrt{p^2+\frac{2}{n+1}\times z^2-p^2-\frac{2z^{n+3}}{n+1}}}$. Put $a=1-p^2$; $v=\frac{n+3}{2}$ and $x=1-z^2$, and it will become $\frac{\frac{1}{2}\sqrt{1-a}\times\dot{x}}{\overline{1-x}\times\sqrt{ax+\frac{1-vx-\overline{1-x}^{v}}{1-v}}}=$

$\frac{1}{2}\times\overline{1-a}^{\frac{1}{2}}$ into $\overline{a-\frac{vx}{2}+\frac{v.\overline{v-2}}{2.3}\times x^2-\frac{v.\overline{v-2}.\overline{v-3}}{2.3.4}\times x^3\,\&c}^{-\frac{1}{2}}\times x^{-\frac{1}{2}}\dot{x}+x^{\frac{1}{2}}\dot{x}+x^{\frac{3}{2}}\dot{x}$ &c.

Now, to find ye Fluent of ye first Term hereof (drawn into ye general Multiplicator) or $\overline{a-\frac{vx}{2}+\frac{v.\overline{v-2}}{2.3}\times x^2\,\&c}^{-\frac{1}{2}}\times x^{-\frac{1}{2}}\dot{x}$, we have (as before) $n=1$, $m=-\frac{1}{2}$, $p=\frac{1}{2}$, $b=\frac{v}{2}$, $\frac{d}{b}=\frac{v-2}{3}$, $\frac{e}{b}=-\frac{v-2.v-3}{3.4}$, &c. Also $R=\frac{v-2}{8}$, $S=\frac{v-2.4v-5}{72}$; and consequently ye Fluent, it self (when ye Body arrives at ye lower Apse)

$=\frac{P}{\sqrt{\frac{1}{2}v}}\times\overline{1+\frac{v-2}{2v}\times a+\frac{5.\overline{v-2}.\overline{4v-5}}{48v^2}\times a^2+\frac{7.\overline{v-2}.\overline{16v^2-37v+22}}{6.48v^3}\,\&c}$.

After ye same Manner the Fluent of ye 2d Term will come out

$=\frac{P}{\sqrt{\frac{1}{2}v}}\times\overline{\frac{a}{v}+\frac{5.\overline{v-2}}{4v^2}\times a^2+\frac{35.\overline{v-2}.\overline{2v-3}}{48v^3}\times a^3\,\&c}$. That of ye 3d

$=\frac{P}{\sqrt{\frac{1}{2}v}}\times\overline{\frac{3a^2}{2v^2}+\frac{35.\overline{v-2}}{12v^3}\times a^3\,\&c}$ &c. &c. Whence, by collecting these several Fluents together, we have

$\frac{P}{\sqrt{\frac{1}{2}v}}\times\overline{1+\frac{1}{2}a+\frac{20v^2-5v+2}{48v^2}\times a^2+\frac{112v^3-63v^2+42v-8}{6.48v^3}\times a^3\,\&c}$ for ye Fluent of ye whole Expression: and this, drawn into $\frac{1}{2}\times\overline{1-\frac{a}{2}-\frac{a^2}{8}\,\&c}$ $(=\frac{1}{2}\times\overline{1-a}^{\frac{1}{2}})$ will be $=\frac{P}{\sqrt{2v}}\times\overline{1\;*\;\frac{v-2.2v-1}{48}\times\frac{a^2}{v^2}+\frac{\overline{v-2.2v-1}^2}{72}\times\frac{a^3}{v^3}\,\&c}$

$=\frac{P}{\sqrt{n+3}}\times\overline{1\;*\;\frac{n-1.n+2}{24}\times\frac{a^2}{\overline{n+3}^2}+\frac{\overline{n-1.n+2}^2}{18}\times\frac{a^3}{\overline{n+3}^3}\,\&c}$. Which, in Degrees, gives $\frac{180}{\sqrt{n+3}}\times\overline{1\;*\;+\frac{n-1.n+2}{24}\times\frac{a^2}{\overline{n+3}^2}+\frac{\overline{n-1.n+2}^2}{18}\times\frac{a^3}{\overline{n+3}^3}\,\&c}$ for ye true Measure of ye Angle required.

May 26. 1748.

Pr. Er. 487. XI.

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Simpson Thomas (1710-1761)

1) L.a.s. 1p. 8° Woolwich 18.V.1748

2) M.a.s. 5p. fol. (18.V.1748)

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